arXiv:1301.4830v1 [math.FA] 21 Jan 2013

MULTIPLICATION AND COMPOSITION OPERATORS BETWEEN TWO DIFFERENT ORLICZ SPACES

Y. ESTAREMI

Abstract. In this paper we consider composition operator C_φ generated by nonsingular measurable transformation $\varphi : \Omega \to \Omega$ and multiplication operator M_u generated by measurable function $u : \Omega \to \mathbb{C}$ between two different Orlicz spaces $L^{\Phi_1}(\Omega, \Sigma, \mu)$ and $L^{\Phi_2}(\Omega, \Sigma, \mu)$, then we investigate boundedness, compactness and essential norm of multiplication and composition operators in term of properties of the mapping φ, the function u and the measure space (Ω, Σ, μ).

1. Introduction and Preliminaries

Let $\Phi : \mathbb{R} \to \mathbb{R}^+$ be a continuous convex function such that
(1)$\Phi(x) = 0$ if and only if $x = 0$.
(2) $\lim_{x\to\infty} \Phi(x) = \infty$.
(3) $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$.

The convex function Φ is called Young's function. With each Young's function Φ, one can associate another convex function $\Psi : \mathbb{R} \to \mathbb{R}^+$ having similar properties, which is defined by

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$

The convex function Ψ is called complementary Young function to Φ. A Young function Φ is said to satisfy the $\triangle_2$ condition (globally) if $\Phi(2x) \leq k\Phi(x),\ x \geq x_0 \geq 0(x_0 = 0)$ for some constant $k > 0$.

If Φ is a Young function, then the set of $\Sigma-$measurable functions

$$L^\Phi(\Sigma) = \{f : \Omega \to \mathbb{C} : \exists k > 0, \int_\Omega \Phi(k|f|)d\mu < \infty\}$$

is a Banach space, with respect to the norm $N_\Phi(f) = \inf\{k > 0 : \int_\Omega \Phi(\frac{f}{k})d\mu \leq 1\}$. $(L^\Phi(\Sigma), N_\Phi(.))$ is called Orlicz space. The usual convergence in the orlicz space $L^\Phi(\Sigma)$ can be introduced in term of the orlicz norm $N_\Phi(.)$ as $u_n \to u$ in $L^\Phi(\Sigma)$ means $N_\Phi(u_n - u) \to 0$. Also, a sequence $\{u_n\}_{n=1}^\infty$ in $L^\Phi(\Sigma)$ is said to converges in Φ-mean to $u \in L^\Phi(\Sigma)$, if

$$\lim_{n\to\infty} I_\Phi(u_n - u) = \lim_{n\to\infty} \int_\Omega \Phi(|u_n - u|)d\mu = 0.$$

Let $\Omega = (\Omega, \Sigma, \mu)$ be a σ-finite complete measure space and let $\varphi : \Omega \to \Omega$ be a measurable transformation, that is, $\varphi^{-1}(A) \in \Sigma$ for any $A \in \Sigma$. If $\mu(\varphi^{-1}(A)) = 0$ for all $A \in \Sigma$ with $\mu(A) = 0$, then φ is said to be nonsingular. This condition means

2000 *Mathematics Subject Classification.* 47B33, 46E30.
Key words and phrases. Composition operator, Multiplication operator, compact operators, Orlicz spaces, essential norm.

that the measure $\mu\circ\varphi^{-1}$, defined by $\mu\circ\varphi^{-1}(A)=\mu(\varphi^{-1}(A))$ for $A\in\Sigma$, is absolutely continuous with respect to the μ (it is usually denoted $\mu\circ\varphi^{-1}\ll\mu$). The Radon-Nikod'ym theorem ensures the existence of a nonnegative locally integrable function h on Ω such that, $\mu\circ\varphi^{-1}(A)=\int_A hd\mu$, $A\in\Sigma$. Any nonsingular measurable transformation φ induces a linear operator (composition operator) C_φ from $L^0(\Omega)$ into itself defined by

$$C_\varphi(f)(t)=f(\varphi(t)) \quad ; t\in\Omega, \quad f\in L^0(\Omega),$$

where $L^0(\Omega)$ denotes the linear space of all equivalence classes of Σ-measurable functions on Ω, that is, we identify any two functions that are equal μ-almost everywhere on Ω. Here the nonsingularity of φ guarantees that the operator C_φ is well defined as a mapping from $L^0(\Omega)$ into itself. If C_φ maps an Orlicz space $L^\Phi(\Omega)$ into itself, then C_φ is called composition operator on $L^\Phi(\Omega)$. Note that, in this case C_φ is bounded.

Let $u:\Omega\to\mathbb{C}$ be a measurable function on Ω. Then the rule taking u to $u.f$, is a linear transformation on $L^0(\Omega)$ and we denote this transformation by M_u. In the case that M_u is continuous, it is called multiplication operator induced by u.

The composition and multiplication operators received considerable attention over the past several decades especially on some measurable function spaces such as L^P-spaces, Bergman spaces and a few ones on Orlicz spaces, such that these operators played an important role in the study of operators on Hilbert spaces.

The basic properties of composition and multiplication operators on measurable function spaces are studied by more mathematicians. For more details on these operators we refer to Abraham [1], Takagi [20], Axler [2], Estaremi and Jabbarzadeh [6], Halmos [7], Lambert [13], Singh and Manhas [16], Takagi [19, 21, 22], Hudzik and Krbec [8], Cui, Hudzik, Kumar and Maligranda [9], Arora [3] and some other works. The multiplication and weighted composition operators are studied on Orlicz spaces in [10, 17]. In the case that φ is an N-function, some results on boundedness of composition operators on Orlicz spaces, are obtained in [11] (see also [14]). As is seen in [18], the essential norm plays an interesting role in the compact problem of concrete operators. Many people have computed the essential norm of various concrete operators. For these studies about composition operators, we refer to [15, 21, 24]. The question of actually calculating the norm and essential norm of a composition and multiplication operators on Orlicz spaces is not a trivial one. In spite of the difficulties associated with computing the essential norm exactly, it is often possible to find upper and lower bound for the essential norm under certain conditions.

In this paper, we are going to present some assertions about boundedness, compactness and essential norm of multiplication and composition operators between two Orlicz spaces. In section 2 we give some necessary and sufficient conditions for boundedness of composition and multiplication operators between two different Orlicz spaces. In section 3 we present some necessary and sufficient conditions for compactness of composition and multiplication operators between two different Orlicz spaces. Then in section 4 by using the compactness assertions, that is proved in section 3, we estimate the essential norm of composition and multiplication operators.

2. **Boundedness**

In this section we present some necessary and sufficient conditions for boundedness of multiplication and composition operators from $L^{\Phi_1}(\Omega)$ into $L^{\Phi_2}(\Omega)$.

Theorem 2.1. Let (Ω, Σ, μ) be a σ-finite nonatomic measure space and $\varphi : \Omega \to \Omega$ be a surjective nonsingular measurable transformation. Denote by h the Radon-Nikodym derivative $\frac{d\mu\circ\varphi^{-1}}{d\mu}$. The the following conditions are equivalent:

(a) The composition operator C_φ is bounded from $L^{\Phi_1}(\Omega)$ into $L^{\Phi_2}(\Omega)$.

(b) The Orlicz space $L^{\Phi_1}(\Omega)$ is embedded continuously into the weighted orlicz space $L_h^{\Phi_2}(\Omega)$.

(c) There are $a, b > 0$ and $g \in L^1(\Omega)$ such that $\Phi_2(au)h(t) \leq b\Phi_1(u) + g(t)$ for all $u > 0$ and $t \in \Omega \setminus A$ with $\mu(A) = 0$.

Proof. $a \to b$. Since φ is surjective, then for all $f \in L^{\Phi_1}(\Omega)$ we have

$$I_{\Phi_2}(C_\varphi(f)) = \int_\Omega \Phi_2(|C_\varphi(f)|)d\mu = \int_{\varphi(\Omega)} h\Phi_2(|f|)d\mu$$
$$= \int_\Omega h\Phi_2(|f|)d\mu = I_{\Phi_2,h}(f).$$

Suppose that (a) is satisfied, then for every $f \in L^{\Phi_1}(\Omega)$ we have

$$N_{\Phi_2,h}(f) = N_{\Phi_2}(C_\varphi(f)) \leq \|C_\varphi\| N_{\Phi_1}(f).$$

This implies that the Orlicz space $L^{\Phi_1}(\Omega)$ is embedded continuously into the weighted orlicz space $L_h^{\Phi_2}(\Omega)$.

By [[12], th 8.5], it is easy to see that $b \to c$.

For $c \to a$, we suppose that (c) holds, then for every $f \in L^{\Phi_1}(\Omega)$ we have

$$I_{\Phi_2}(\frac{aC_\varphi(f)}{N_{\Phi_1}(f)}) = \int_\Omega \Phi_2(\frac{af(t)}{N_{\Phi_1}(f)})h(t)d\mu$$
$$\leq b\int_\Omega \Phi_1(\frac{f(t)}{N_{\Phi_1}(f)})d\mu + \int_\Omega g(t)d\mu \leq b + \int_\Omega g(t)d\mu \leq M',$$

where $M' > 1$. This implies that $I_{\Phi_2}(\frac{(aC_\varphi(f))}{M' N_{\Phi_1}(f)}) \leq 1$, thus $N_{\Phi_2}(C_\varphi(f)) \leq \frac{M'}{a} N_{\Phi_1}(f)$.

Theorem 2.2. If $C_\varphi : L^{\Phi_1}(\Omega) \to L^{\Phi_2}(\Omega)$ is a linear transformation, then C_φ is bounded.

Proof. By applying closed graph theorem, injectivity of Φ_1 and Φ_2 and the fact that the norm-convergence implies the Φ-mean-convergence, we conclude that C_φ is bounded.

Remark.2.3 By theorem 2.2 we have: $C_\varphi \in B(L^{\Phi_1}, L^{\Phi_2})$ if and only if $C_\varphi(L^{\Phi_1}) \subseteq L^{\Phi_2}$. Thus the following conditions are equivalent:

(a) The composition operator C_φ is bounded from $L^{\Phi_1}(\Omega)$ into $L^{\Phi_2}(\Omega)$.

(b) For every $f \in L^{\Phi_1}(\Omega)$, there exists $\lambda > 0$ such that

$$\int_\Omega h\Phi_2(\lambda|f|)d\mu < \infty.$$

(c) The Orlicz space $L^{\Phi_1}(\Omega)$ is embedded continuously into the weighted orlicz space $L_h^{\Phi_2}(\Omega)$.

(d) There are $a, b > 0$ and $g \in L^1(\Omega)$ such that $\Phi_2(au)h(t) \leq b\Phi_1(u) + g(t)$ for all $u > 0$ and $t \in \Omega \setminus A$ with $\mu(A) = 0$.

Theorem 2.4. Let $u : \Omega \to \mathbb{C}$ be a measurable function. Then

(a) If there exists $M > 0$ such that

$$\Phi_2(|u(x)\alpha|) \leq \Phi_1(M|\alpha|)$$

for μ-almost all $x \in \Omega$ and $\alpha \in \mathbb{C}$, Then $M_u : L^{\Phi_1}(\Omega) \to L^{\Phi_2}(\Omega)$ is a bounded operator.

(b) If (Ω, Σ, μ) is non-atomic measure space and the operator $M_u : L^{\Phi_1}(\Omega) \to L^{\Phi_2}(\Omega)$ is bounded, then there exists $M > 0$ such that

$$\Phi_2(u(x)\alpha) \leq \Phi_1(M\alpha)$$

for μ-almost all $x \in \Omega$ and $\alpha \in \mathbb{C}$.

Proof. (a) For every $f \in L^{\Phi_1}(\Omega)$

$$\int_\Omega \Phi_2(\frac{uf}{MN_{\Phi_1(f)}})d\mu \leq \int_\Omega \Phi_1(\frac{Mf}{MN_{\Phi_1(f)}})d\mu \leq 1.$$

Therefore $N_{\Phi_2}(M_u(f)) \leq MN_{\Phi_1(f)}$, for every $f \in L^{\Phi_1}(\Omega)$.

(b) If the condition of theorem is not satisfied, then for every $n \in \mathbb{N}$ there exists a measurable set F_n of Ω and some $\alpha_n \in \mathbb{C}$ such that

$$F_n = \{x \in \Omega : \Phi_2(|u(x)\alpha_n|) > \Phi_1(2^n n^2 \alpha_n)\}$$

is a measurable set of positive measure. Since μ is non-atomic, we can choose a disjoint sequence of measurable sets $\{E_n\}$ such that $E_n \subseteq F_n$ and

$$\mu(E_n) = \frac{\Phi_2(|\alpha_1|)}{2^n \Phi_1(n^2|\alpha_n|)}.$$

Let $f = \Sigma_{n=1}^{\infty} c_n \chi_{E_n}$, where $c_n = n|\alpha_n|$. Suppose that$\alpha > 0$ and $n_0 > \alpha$, then

$$\int_\Omega \Phi_1(\alpha f)d\mu = \Sigma_{n=1}^{\infty} \int_\Omega \Phi_1(\alpha c_n)\chi_{E_n} d\mu$$

$$= \Sigma_{n=1}^{n_0} \Phi_1(\alpha c_n)\mu(E_n) + \Sigma_{n=n_0+1}^{\infty} \frac{\Phi_1(\alpha c_n)\Phi_2(|\alpha_1|)}{2^n \Phi_1(n^2|\alpha_n|)}$$

$$\leq \Sigma_{n=1}^{n_0} \Phi_1(\alpha c_n)\mu(E_n) + \Sigma_{n=n_0+1}^{\infty} \frac{\Phi_1(n^2|\alpha_n|)\Phi_2(|\alpha_1|)}{2^n \Phi_1(n^2|\alpha_n|)} < \infty.$$

If $\alpha > \frac{1}{n_0}$, then

$$\int_\Omega \Phi_2(\alpha|u.f|)d\mu \geq \Sigma_{n \geq n_0} \int_{E_n} \Phi_2(\alpha n|u\alpha_n|)d\mu \geq$$

$$\Sigma_{n \geq n_0} \int_{E_n} \Phi_2(|u\alpha_n|)d\mu \geq \Sigma_{n \geq n_0} \Phi_1(2^n n^2|\alpha_n|)\mu(E_n)$$

$$\geq \Sigma_{n \geq n_0} \Phi_2(|\alpha_1) = \infty.$$

This is a contradiction.

Theorem 2.5. If $M_u : L^{\Phi_1}(\Omega) \to L^{\Phi_2}(\Omega)$ is a linear transformation, then M_u is bounded.

Proof. By applying closed graph theorem, injectivity of Φ_1 and Φ_2 and the fact that the norm-convergence implies the Φ-mean-convergence, we conclude that C_φ is bounded.

3. Compactness

In this section we present some necessary and sufficient condition for composition and multiplication operators to be compact. Recall that an atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space (Ω, Σ, μ) with no atoms is called non-atomic measure space. It is well-known fact that every σ-finite measure space (Ω, Σ, μ) can be partitioned uniquely as $\Omega. = B \cup \{A_j : j \in \mathbb{N}\}$, where $\{A_j\}_{j\in\mathbb{N}}$ is a countable collection of pairwise disjoint atoms and $B \in \Sigma$, being disjoint from each A_j, is non-atomic (see [23]). Since Σ is σ-finite, so $a_j := \mu(A_j) < \infty$, for all $j \in \mathbb{N}$. A bounded linear operator $T : E \to E$ (where E is a Banach space) is called compact, if $T(B_1)$ has compact closure, where B_1 denotes the closed unit ball of E.

Theorem 3.1. Let $T = C_\varphi$ be bounded from $L^{\Phi_1}(\Omega, \Sigma, \mu)$ to $L^{\Phi_2}(\Omega, \Sigma, \mu)$. Then C_φ is compact if and only if $N_\varepsilon = \{x \in \Omega : \Phi_2(|\alpha|)h(t) > \Phi_1(\varepsilon|\alpha|), \alpha \in \mathbb{C}\}$ consists of finitely many atoms, for all $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$ and $N_\varepsilon = \cup_{i=1}^{n} C_n$ consists of finitely many atoms. Put $T_\varepsilon = C_\varphi M_{\chi_{N_\varepsilon}}$. Since $\Sigma-$measurable functions are constant on $\Sigma-$atoms and (Ω, Σ, μ) is $\sigma-$finite, then $M_{\chi_{N_\varepsilon}}$ is a compact operator on $L^{\Phi_1}(\Omega, \Sigma, \mu)$. Thus the operator $T_\varepsilon = C_\varphi M_{\chi_{N_\varepsilon}}$ is compact from $L^{\Phi_1}(\Omega, \Sigma, \mu)$ to $L^{\Phi_2}(\Omega, \Sigma, \mu)$. Hence for every $f \in L^{\Phi_1}(\Omega, \Sigma, \mu)$

$$\int_\Omega \Phi_2\left(\frac{(T - T_\varepsilon)(f)}{\varepsilon N_{\Phi_1}(f)}\right)d\mu = \int_\Omega \Phi_2\left(\frac{C_\varphi(\chi_{\Omega\setminus N_\varepsilon} f)}{\varepsilon N_{\Phi_1}(f)}\right)d\mu$$
$$= \int_{\Omega\setminus N_\varepsilon} h\Phi_2\left(\frac{f}{\varepsilon N_{\Phi_1}(f)}\right)d\mu \le \int_{\Omega\setminus N_\varepsilon} \Phi_1\left(\frac{\varepsilon f}{\varepsilon N_{\Phi_1}(f)}\right)d\mu$$
$$= \int_{\Omega\setminus N_\varepsilon} \Phi_1\left(\frac{f}{N_{\Phi_1}(f)}\right)d\mu \le 1.$$

This implies that $N_{\Phi_2}(Tf - T_\varepsilon f) \le \varepsilon N_{\Phi_1}(f)$. Thus T is compact.

Conversely, suppose there exists $\varepsilon > 0$ such that N_ε consists of infinitely many atoms or a non-atomic subset of positive measure. In both cases we can find a sequence $\{B_n\}_{n\in\mathbb{N}}\}$ of disjoint measurable subsets of N_ε with $0 < \mu(B_n) < \infty$. Put $f_n = \frac{\chi_{B_n}}{N_{\Phi_1}(\chi_{B_n})}$. Hence

$$\int_\Omega \Phi_1\left(\frac{\varepsilon f_n}{N_{\Phi_2}(f_n \circ \varphi)}\right)d\mu \le \int_\Omega h\Phi_2\left(\frac{f_n}{N_{\Phi_2}(f_n)}\right)d\mu$$
$$= \int_\Omega \Phi_2\left(\frac{f_n \circ \varphi}{N_{\Phi_2}(f_n \circ \varphi)}\right)d\mu \le 1.$$

So $\varepsilon = N_{\Phi_1}(\varepsilon f_n) \le N_{\Phi_2}(f_n \circ \varphi)$. Since B_n's are disjoint. for $n \ne m$

$$N_{\Phi_2}(f_n \circ \varphi - f_m \circ \varphi) \ge N_{\Phi_2}(f_n \circ \varphi) \ge \varepsilon.$$

So $\{f_n \circ \varphi\}_{n\in\mathbb{N}}$ has no convergent subsequence. This mean's $T = C_\varphi$ can not be compact.

Corollary 3.2. If (Ω, σ, μ) is nonatomic measure space, then there is not nonzero compact composition operator between $L^{\Phi_1}(\Omega, \Sigma, \mu)$ and $L^{\Phi_2}(\Omega, \Sigma, \mu)$.

Theorem 3.3. Let M_u be bounded from $L^{\Phi_1}(\Omega, \Sigma, \mu)$ to $L^{\Phi_2}(\Omega, \Sigma, \mu)$. Then M_u is compact if and only if $N_\varepsilon(u) = \{x \in \Omega : \Phi_2(|u(x)\alpha|) > \Phi_1(\varepsilon|\alpha|), \alpha \in \mathbb{C}\}$ consists of finitely many atoms, for all $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$ and $N_\varepsilon = N_\varepsilon(u) = \cup_{i=1}^n C_n$ consists of finitely many atoms. Put $u_\varepsilon = u\chi_{N_\varepsilon}$ and $T_\varepsilon = M_{u_\varepsilon}$. Since $\Sigma-$measurable functions are constant on $\Sigma-$atoms and (Ω, Σ, μ) is $\sigma-$finite. We have

$$M_{u_\varepsilon}(f) = \sum_{i=1}^n u(C_i)f(C_i)\chi_{C_i} \in \{\sum_{i=1}^n \alpha_i\chi_{C_i} : \alpha_i \in \mathbb{C}\} \subseteq L^{\Phi_2}(\Omega, \Sigma, \mu).$$

Thus M_{u_ε} is finite rank. Hence for every $f \in L^{\Phi_1}(\Omega, \Sigma, \mu)$

$$\int_\Omega \Phi_2(\frac{(u-u_\varepsilon)f}{\varepsilon N_{\Phi_1}(f)})d\mu = \int_{\Omega\setminus N_\varepsilon} \Phi_2(\frac{uf}{\varepsilon N_{\Phi_1}(f)})d\mu$$
$$\leq \int_{\Omega\setminus N_\varepsilon} \Phi_1(\frac{\varepsilon f}{\varepsilon N_{\Phi_1}(f)})d\mu = \int_{\Omega\setminus N_\varepsilon} \Phi_1(\frac{f}{N_{\Phi_1}(f)})d\mu \leq 1.$$

This implies that $N_{\Phi_2}(M_u f - M_{u_\varepsilon} f) \leq \varepsilon N_{\Phi_1}(f)$. Thus M_u is compact.

Conversely, suppose there exists $\varepsilon > 0$ such that N_ε consists of infinitely many atoms or a non-atomic subset of positive measure. In both cases we can find a sequence $\{B_n\}_{n\in\mathbb{N}}\}$ of disjoint measurable subsets of N_ε with $0 < \mu(B_n) < \infty$. Put $f_n = \frac{\chi_{B_n}}{N_{\Phi_1}(\chi_{B_n})}$. Hence

$$\int_\Omega \Phi_1(\frac{\varepsilon f_n}{N_{\Phi_2}(uf_n)})d\mu \leq \int_\Omega \Phi_2(\frac{uf_n}{N_{\Phi_2}(uf_n)})d\mu \leq 1.$$

So $\varepsilon = N_{\Phi_1}(\varepsilon f_n) \leq N_{\Phi_2}(uf_n)$. Since B_n's are disjoint. for $n \neq m$ $N_{\Phi_2}(uf_n - uf_m) \geq N_{\Phi_2}(uf_n) \geq \varepsilon$. So $\{uf_n\}_{n\in\mathbb{N}}$ has no convergent subsequence. This mean's M_u can not be compact.

Corollary 3.4. If (Ω, σ, μ) is nonatomic measure space, then there is not nonzero compact multiplication operator between $L^{\Phi_1}(\Omega, \Sigma, \mu)$ and $L^{\Phi_2}(\Omega, \Sigma, \mu)$.

4. **Essential norm**

Let $\mathfrak{B}$ be a Banach space and $\mathcal{K}$ be the set of all compact operators on $\mathfrak{B}$. For $T \in L(\mathfrak{B})$, the Banach algebra of all bounded linear operators on $\mathfrak{B}$ into itself, the essential norm of T means the distance from T to $\mathcal{K}$ in the operator norm, namely $\|T\|_e = \inf\{\|T - S\| : S \in \mathcal{K}\}$. Clearly, T is compact if and only if $\|T\|_e = 0$. As is seen in [18], the essential norm plays an interesting role in the compact problem of concrete operators.

Theorem 4.1. Let $\varphi : \Omega \to \Omega$ be nonsingular measurable transformation and let $T = C_\varphi : L^{\Phi_1}(\Omega, \Sigma, \mu) \to L^{\Phi_2}(\Omega, \Sigma, \mu)$. If $\beta_1 = \inf\{\varepsilon > 0 : N_\varepsilon$ consists of finitely many atoms$\}$. Then

(a) $\|C_\varphi\|_e \leq \beta_1$.

(b) Let $\Phi_1 \in \triangle_2$ and $\mu(C_n) \to 0$ or $\{\mu(C_n)\}_{n\in\mathbb{N}}$ has no convergent subsequence. Then $\beta_1 \leq \|C_\varphi\|_e$.

Proof. (a) Let $\varepsilon > 0$. Then $N_{\varepsilon+\beta_1}$ consist of finitely many atoms. Put $T_{\varepsilon+\beta_1} = C_\varphi M_{\chi_{N_{\varepsilon+\beta_1}}}$. So $T_{\varepsilon+\beta_1}$ is compact. Also for $f \in L^{\Phi_1}(\Omega, \Sigma, \mu)$

$$\int_\Omega \Phi_2(\frac{(T - T_{\varepsilon+\beta_1})(f)}{(\varepsilon+\beta_1)N_{\Phi_1}(f)})d\mu = \int_\Omega \Phi_2(\frac{C_\varphi(\chi_{\Omega\setminus N_{\varepsilon+\beta_1}} f)}{(\varepsilon+\beta_1)N_{\Phi_1}(f)})d\mu$$

$$= \int_{\Omega\setminus N_{\varepsilon+\beta_1}} h\Phi_2(\frac{f}{(\varepsilon+\beta_1)N_{\Phi_1}(f)})d\mu \leq \int_{\Omega\setminus N_{\varepsilon+\beta_1}} \Phi_1(\frac{(\varepsilon+\beta_1)f}{(\varepsilon+\beta_1)N_{\Phi_1}(f)})d\mu$$

$$= \int_{\Omega\setminus N_{\varepsilon+\beta_1}} \Phi_1(\frac{f}{N_{\Phi_1}(f)})d\mu \leq 1.$$

This implies that $N_{\Phi_2}(Tf - T_{\varepsilon+\beta_1}f) \leq (\varepsilon+\beta_1)N_{\Phi_1}(f)$. Hence

$$\|T\|_e \leq \|T - T_{\varepsilon+\beta_1}\| \leq \varepsilon + \beta_1.$$

Thus $\|T\|_e \leq \beta_1$.

(b) Let $0 < \varepsilon < \beta_1$. Then by definition, $N_{\beta_1-\varepsilon}(u)$ contains infinitely many atoms or a non- atomic subset of positive measure. If $N_{\beta_1-\varepsilon}(u)$ consists a non- atomic subset, then we can find a sequence $\{B_n\}_{n\in\mathbb{N}}$ such that $\mu(B_n) < \infty$ and $\mu(B_n) \to 0$. Put $f_n = \frac{\chi_{B_n}}{N_{\Phi_1}(\chi_{B_n})}$, then for every $A \in \Sigma$ with $0 < \mu(A) < \infty$ we have

$$\int_\Omega f_n \chi_A d\mu = \mu(A \cap B_n)\Phi_1^{-1}(\frac{1}{\mu(B_n)}) \leq \frac{\Phi_1^{-1}(\frac{1}{\mu(B_n)})}{\frac{1}{\mu(B_n)}} \to 0.$$

when $n \to \infty$. Also, if $N_{\beta_1-\varepsilon}(u)$ consists infinitely many atoms $\{C'_n\}_{n\in\mathbb{N}}$. We set $f_n = \frac{\chi_{C'_n}}{N_{\Phi_1}(\chi_{C'_n})}$. Then for every $A \in \Sigma$ with $0 < \mu(A) < \infty$ we have

$$\int_\Omega f_n \chi_A d\mu = \mu(A \cap C'_n)\Phi_1^{-1}(\frac{1}{\mu(C'_n)}).$$

If $\{\mu(C_n)\}_{n\in\mathbb{N}}$ has no convergent subsequence, then there exists n_0 such that for $n > n_0$, $\mu(A \cap C'_n) = 0$ and if $\mu(C_n) \to 0$ then $\mu(C'_n) \to 0$. Thus $\int_\Omega f_n\chi_A d\mu = \mu(A\cap C'_n)\Phi_1^{-1}(\frac{1}{\mu(C'_n)}) \to 0$ in both cases. These imply that $f_n \to 0$ weakly. So

$$\int_\Omega \Phi_1(\frac{(\beta_1-\varepsilon)f_n}{N_{\Phi_2}(f_n\circ\varphi)})d\mu \leq \int_\Omega \Phi_2(\frac{f_n\circ\varphi}{N_{\Phi_2}(f_n\circ\varphi)})d\mu.$$

Thus $N_{\Phi_2}(C_\varphi(f_n \circ \varphi)) \geq \beta_1 - \varepsilon$.

Also, there exists compact operator $T \in L(L^{\Phi_1}(\Omega, \Sigma, \mu), L^{\Phi_1}(\Omega, \Sigma, \mu))$ such that $\|C_\varphi\|_e \geq \|C_\varphi - T\| - \varepsilon$. Hence $N_{\Phi_2}(Tf_n) \to o$ and so there exists $N > 0$ such that for each $n > N$, $N_{\Phi_2}(Tf_n) \leq \varepsilon$. So

$$\|C_\varphi\|_e \geq \|C_\varphi - T\| - \varepsilon \geq |N_{\Phi_2}(f_n\circ\varphi) - N_{\Phi_2}(Tf_n)| \geq \beta_1 - \varepsilon - \varepsilon,$$

thus we conclude that $\|C_\varphi\|_e \geq \beta_1$.

Theorem 4.2. Let $u : \Omega \to \mathbb{C}$ be $\Sigma-$measurable and Let $M_u : L^{\Phi_1}(\Omega, \Sigma, \mu) \to L^{\Phi_2}(\Omega, \Sigma, \mu)$. If $\beta_2 = \inf\{\varepsilon > 0 : N_\varepsilon$ consists of finitely many atoms$\}$. Then

(a) $\|M_u\|_e \leq \beta_2$.

(b) Let $\Phi_1 \in \triangle_2$ and $\mu(C_n) \to 0$ or $\{\mu(C_n)\}_{n\in\mathbb{N}}$ has no convergent subsequence. Then $\beta_2 \leq \|M_u\|_e$.

Proof. (a) Let $\varepsilon > 0$. Then $N_{\varepsilon+\beta_2}$ consist of finitely many atoms. Put $u_{\varepsilon+\beta_2} = u\chi_{N_{\varepsilon+\beta_2}}$ and $M_{u_{\varepsilon+\beta_2}}$. So $M_{u_{\varepsilon+\beta_2}}$ is finite rank and so compact. Also for $f \in L^{\Phi_1}(\Omega, \Sigma, \mu)$

$$\int_\Omega \Phi_2(\frac{(u-u_{\varepsilon+\beta_2})f}{(\varepsilon+\beta_2)N_{\Phi_1}(f)})d\mu = \int_{\Omega\setminus N_{\varepsilon+\beta_2}} \Phi_2(\frac{uf}{(\varepsilon+\beta_2)N_{\Phi_1}(f)})d\mu$$
$$\leq \int_{\Omega\setminus N_{\varepsilon+\beta_2}} \Phi_1(\frac{f}{N_{\Phi_1}(f)})d\mu \leq 1.$$

Hence $N_{\Phi_2}(uf - u_{\varepsilon+\beta_2}f) \leq (\varepsilon+\beta_2)N_{\Phi_1}(f)$ and so

$$\|M_u\|_e \leq \|M_u - M_{u_{\varepsilon+\beta_2}}\| \leq \varepsilon + \beta_2.$$

Thus $\|M_u\|_e \leq \beta_2$.

(b) Let $0 < \varepsilon < \beta_2$. Then by definition, $N_{\beta_2-\varepsilon}(u)$ contains infinitely many atoms or a non- atomic subset of positive measure. If $N_{\beta_2-\varepsilon}(u)$ consists a non- atomic subset, then we can find a sequence $\{B_n\}_{n\in\mathbb{N}}$ such that $\mu(B_n) < \infty$ and $\mu(B_n) \to 0$. Put $f_n = \frac{\chi_{B_n}}{N_{\Phi_1}(\chi_{B_n})}$, then for every $A \in \Sigma$ with $0 < \mu(A) < \infty$ we have

$$\int_\Omega f_n\chi_A d\mu = \mu(A\cap B_n)\Phi_1^{-1}(\frac{1}{\mu(B_n)}) \leq \frac{\Phi_1^{-1}(\frac{1}{\mu(B_n)})}{\frac{1}{\mu(B_n)}} \to 0.$$

when $n \to \infty$. Also, if $N_{\beta_2-\varepsilon}(u)$ consists infinitely many atoms $\{C'_n\}_{n\in\mathbb{N}}$. We set $f_n = \frac{\chi_{C'_n}}{N_{\Phi_1}(\chi_{C'_n})}$. Then for every $A \in \Sigma$ with $0 < \mu(A) < \infty$ we have

$$\int_\Omega f_n\chi_A d\mu = \mu(A\cap C'_n)\Phi_1^{-1}(\frac{1}{\mu(C'_n)}).$$

If $\{\mu(C_n)\}_{n\in\mathbb{N}}$ has no convergent subsequence, then there exists n_0 such that for $n > n_0$, $\mu(A\cap C'_n) = 0$ and if $\mu(C_n) \to 0$ then $\mu(C'_n) \to 0$. Thus $\int_\Omega f_n\chi_A d\mu = \mu(A\cap C'_n)\Phi_1^{-1}(\frac{1}{\mu(C'_n)}) \to 0$ in both cases. These imply that $f_n \to 0$ weakly. So

$$\int_\Omega \Phi_1(\frac{(\beta-\varepsilon)f_n}{N_{\Phi_2}(uf_n)})d\mu \leq \int_\Omega \Phi_2(\frac{uf_n}{N_{\Phi_2}(uf_n)})d\mu.$$

Thus $N_{\Phi_2}(uf_n) \geq \beta_2 - \varepsilon$.

Also, there exists compact operator $T \in L(L^{\Phi_1}(\Omega,\Sigma,\mu), L^{\Phi_2}(\Omega,\Sigma,\mu))$ such that $\|M_u\|_e \geq \|T - M_u\| - \varepsilon$. Hence $N_{\Phi_2}(Tf_n) \to o$ and so there exists $N > 0$ such that for each $n > N$, $N_{\Phi_2}(Tf_n) \leq \varepsilon$. So

$$\|M_u\|_e \geq \|M_u - T\| - \varepsilon \geq |N_{\Phi_2}(uf_n) - N_{\Phi_2}(Tf_n)| \geq \beta_2 - \varepsilon - \varepsilon,$$

thus we conclude that $\|M_u\|_e \geq \beta_2$.

Corollary 4.3. If $\mu(\Omega) < \infty$ and $\Phi_1 \in \triangle_2$. Then

(a) $\|C_\varphi\|_e = \beta_1$.

(b) $\|M_u\|_e = \beta_2$.

Corollary 4.4. If $\mu(\Omega) < \infty$ and $\Phi_1 \in \triangle_2$. Then

(a) C_φ is compact if and only if $\beta_1 = 0$.

(b) M_u is compact if and only if $\beta_2 = 0$.

Example 4.5. (a) Let $\Omega = \mathbb{N}$, μ be counting measure on Ω and φ be injective transformation on Ω. If we set $\Phi_1(n) = \frac{n^3}{3}$ and $\Phi_2(n) = \frac{n^2}{2}$, for all $n \in \mathbb{N}$. Then by using theorem 3.1 the operator C_φ is compact from $L^{\Phi_1}(\mathbb{N}, \Sigma, \mu)$ into $L^{\Phi_2}(\mathbb{N}, \Sigma, \mu)$. Also, C_φ is not compact from $L^{\Phi_2}(\mathbb{N}, \Sigma, \mu)$ into $L^{\Phi_1}(\mathbb{N}, \Sigma, \mu)$. Also, if $u(n) = \frac{n^2}{n+1}$, then by theorem 3.3 the multiplication operator M_u is not compact from $L^{\Phi_1}(\mathbb{N}, \Sigma, \mu)$ into $L^{\Phi_2}(\mathbb{N}, \Sigma, \mu)$ and if $u(n) = \frac{1}{n^2}$, then by theorem 3.3 the multiplication operator M_u is compact from $L^{\Phi_1}(\mathbb{N}, \Sigma, \mu)$ into $L^{\Phi_2}(\mathbb{N}, \Sigma, \mu)$.

(b) Let $\Omega = [0, 1) \cup (\mathbb{N} - \{1\})$, where $\mathbb{N}$ is the set of natural numbers. Let μ be the Lebesque measure on $[0, 1)$ and $\mu(\{n\}) = 1$, if $n \in (\mathbb{N} - \{1\})$. If we set $\Phi_1(x) = e^x - x - 1$, $\Phi_2(n) = \frac{x^5}{5}$ and $u(x) = x^2 + 2$ for $x \in \Omega$, then then by theorem 3.3 the multiplication operator M_u is not compact from $L^{\Phi_1}(\mathbb{N}, \Sigma, \mu)$ into $L^{\Phi_2}(\mathbb{N}, \Sigma, \mu)$.

ACKNOWLEDGEMENT The author would like to thank the professor Romesh Kumar for very helpful comments and valuable suggestions.

References

[1] M.B. Abrahamse , Multiplication Operators, Lecture Notes in Mathematics, **693**, (1978), 17-36.

[2] A. Axler, Multiplication Operators on Bergman Spaces, J. Reine,. Angew Math. **336** (1982), 26-44.

[3] S. C. Arora, Gopal Datt and Satish Verma, Composition operators on Lorentz spaces, Bull. Austral. Math. Soc. **76** (2007), 205-214.

[4] Cui, Yunan; Hudzik, Henryk; Kumar, Romesh; Maligranda, Lech Composition operators in Orlicz spaces. J. Aust. Math. Soc. 76 (2004), no. 2, 189206.

[5] R.G. Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, New York, 1972.

[6] Y. Estaremi and M.R. Jabbarzadeh, Weighted lambert type operators on L^p-spaces, Oper. and Matrices. **7** (2013), 101-116.

[7] P.R. Halmos, A Hilbert Space Problem Book, Van Nostr and, Princeton, N.J., 1961.

[8] H. Hudzik and M. Krbec, On non-effective weights in Orlicz spaces. Indag. Math. (N.S.) **18** (2007), 215-231.

[9] Y. Cui, H. Hudzik, R. Kumar and L. Maligranda, Composition operators in Orlicz spaces, J. Aust. Math. Soc. **76** (2004), 189-206.

[10] B.S. Komal AND S. Gupta, Multiplication operators between Orlicz spaces, Integral equation and operator theory. **41** (2001), 324-330.

[11] R. Kumar, Composition operators on Orlicz spaces, Integral Equations Operator Theory **29** (1997), 1722.

[12] J. Musielak, Orlicz spaces and modular spaces, Lecture Notes in Math.1034 (Springer, Berlin, 1983).

[13] A. Lambert, Operator algebras related to measure preserving transformations of finite order, Rocky Mountain J. Math. **14** (1984), 341-349
[14] M. M. Rao, Convolutions of vector fieldsII: random walk models, Nonlinear Anal, Theory Methods Appl. **47** (2001), 3599-3615.
[15] Montes-Rodrguez, A. The essential norm of a composition operator on Bloch spaces, Pacific J. Math. 188, 339-351 (1999).
[16] R. K. Singh and J. S. Manhas, Composition operators on function spaces, North Holland Math. Studies 179, Amsterdam 1993.
[17] S. Gupta, B. S Komal and N. Suri, Weighted composition operators on Orlicz spaces, Int. J. Contemp. Math. Sciences. **1**, 11-20 (2010).
[18] J. H. Shapiro, The essential norm of a composition operator, Analls of Math. **125** (1987), 375-404.
[19] H. Takagi, Compact weighted composition operators on L^p, Proc. Amer. Math. Soc. **116** (1992), 505-511.
[20] H. Takagi, FredhoIm Weighted Composition Operators, Integral Equations and Operator Theory **16** (1993), 267-276.
[21] Takagi, T., Miura, T. and Takahasi, Sin-Ei: Essential norm and stability constants of weighted composition operators on $C(X)$, Bull. Korean Math. Soc. **40**, 583-591 (2003).
[22] H. Takagi and K. Yokouchi, Multiplication and composition operators between two L^p-spaces, Contemporary Math. **232**(1999), 321-338.
[23] A. C. Zaanen, Integration, 2nd ed., North-Holland, Amsterdam, 1967.
[24] Zheng, L.: The essential norms and spectra of composition operators on H^∞, Pacific J. Math. **203**, 503-510 (2002).

Y. Estaremi

Department of Mathematics, Payame Noor University (PNU), P. O. Box: 19395-3697, Tehran- Iran
E-mail address: estaremi@gmail.com